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Aggregate Dynamics of Lumpy Agents

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AGGREGATE DYNAMICS OF LUMPY AGENTS

ABSTRACT

This paper identifies the criteria for dynamic synchronization of the movement of agents who make intermittent adjustment to inventory stocks, leading to "harmonic resonance" rather than cancellation. I use a discrete Markov process model of (S,s) inventory adjustment to establish a theoretical framework for the aggregate dynamics and use simulations to demonstrate the distribution effects of a discrete model of lumpy behavior. The paper identifies circumstances that lead to increased skewness of the distribution of agents over the inventory interval. This has application in financial, labor and commodity markets.

JEL CLASSIFICATION: E1, E2

KEYWORDS: Inventory, (S,s), Aggregation, Dynamics

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1. Introduction

That economic agents make lumpy decisions is intuitively acceptable. A particularly nettlesome question is whether or not this individual lumpiness matters in the aggregate. The short answer is: "Sometimes." Intuitively we expect a smooth aggregate of many lumpy agents. In a dynamic sense however, intermittent adjustment epitomizes non-linearity in that initial conditions matter and small perturbations can cause large changes. For example, in the case of (S,s) inventory behavior, if on the one hand, many agents end up near the trigger point for replenishment as a result of idiosyncratic shocks, then a small positive aggregate shock to demand can lead to a large increase in total orders. If, on the other hand, many agents are close to maximum inventory capacity, a large positive shock to demand may induce only a small contemporary change in orders as inventory provides the buffer.

The appropriate question to ask then, is "when do we need to be concerned about intermittent or (S,s) adjustment in the aggregate?" Obviously, if there is sufficient negative correlation between individual agents, then the aggregate will have little resemblance to the individual characteristics. Positive correlations between individual agents will produce exaggerated aggregate response. It should be sufficient therefore to identify the conditions which are likely to induce negative or positive correlation in individual agents. In any event it helps to know what circumstances will lead to increased skewness of the distribution of agents over the inventory interval.

Tackling the intractability of aggregation of agents making intermittent adjustments in their portfolios (whether of financial, labor, or commodity market goods) has led to even more intractable mathematics. This paper hopefully will strike a balance between the mathematical complexity and the simulation simplification.

I will first review the motivation for S,s behavior, and second use two methods to assess the aggregate dynamics of agents who make intermittent adjustments in the presence of both aggregate and idiosyncratic shocks. The next section discusses some of the recent work in the area. Section 3 reviews the theoretical justification for (S,s) behaviour. Section 4 presents a Markov model of (S,s) behavior and shows the steady state behaviour of a two agent economy facing aggregate shocks. Section 5 presents the results of simulations of (S,s) agents subject to both aggregate and idiosyncratic shocks. A conclusion and summary are presented in section 6.

2. Literature Review

Caplin (1985) provides a general theory of the aggregation of agents who use (S,s) inventory policies, demonstrating that the variance of orders will exceed the variance of sales under these circumstances. Caplin models inventory as a Markov process where the future demand of a given retailer is drawn independently from a probability distribution. Therefore, aggregate inventory moves from state to state as a function of positively correlated demand or negatively correlated demand (aggregate or "zero-sum" idiosyncratic movement). Using this Markov model in continuous time, he concludes that in the long run the inventory levels of individual retailers are mutually independent, regardless of the correlation in sales. The implication is that there can be no induced dependencies of the movement in inventories from correlation in sales. This is the result which allows characterization of the aggregate implications of (S,s) policies.

Caballero and Engle (1991) provide a framework for analyzing the aggregate dynamics of (S,s) economies. In particular, Caballero and Engle state conditions under which (S,s) economies achieve a steady state, where steady state is defined as a condition in which the distribution of inventories are invariant to the distribution of demand. One of these conditions states that if the agents' inventory are initially uniformly distributed in the (S,s) interval, and if sales are subject to aggregate (correlated) shocks, the inventories will remain uniformly distributed on the (S,s) interval. This coincides with Caplin's observation regarding independence. However, both Caplin (1985) and Caballero and Engle (1991) use a continuous time model, where agents are able to adjust when inventory is exactly equal to s (as compared to less than or equal to s). This implies that having started out uniformly distributed, each agent will arrive at s at a unique time t.

If we assume time is discrete however, we can induce positive correlation without any other assumption. For example, If we assume that firms cannot make orders over the weekend, then it becomes easy to see that firms who surpass their trigger point for inventory replenishment at different times during a weekend would coordinate orders on Monday. In this sense, firms are moved closer together in the interval [S,s) simply because time is discrete. Bar-Ilan and Blinder (1992) use an (S,s) model for durable goods consumption. They assume agents replenish after goods depreciate to some trigger level. In their example of the depreciation of cars to a minimum before repurchase we can think of a severe winter in say, Atlanta, Georgia, causing many accidents simultaneously, as a shock to the depreciation rate, resulting in a large increase in new car purchases.

Exclusively idiosyncratic shocks would tend to make the distance between any two agents' position in the inventory interval a random variable. Aggregate shocks would tend to maintain the distance between firms in continuous time. In discrete time however, aggregate shocks may tend to lower the distance between agents by pushing several agents below the minimum simultaneously. The important difference between the continuous and the discrete-time model is that in the discrete-time model the interval is open at s i.e. (s,S] whereas, in continuous time it is closed at s, i.e [s,S]. There is a finite probability of having agents who are separated by δ_k on the interval end up together in the interval [0,s] after a shock greater than δ_k. Once firms end up in the interval together, they replenish together and remain together absent idiosyncratic shocks. Heterogeneity of firms can also alter the relative distances between agents in the (S,s) interval in the same way that idiosyncratic shocks do. When heterogeneous firms synchronize, aggregate shocks can separate them by pushing them to replenishment at different times. Changes in individual or aggregate parameters such as the variance of demand, might result in changes in the upper bound which could also shuffle the relative location of firms after a replenishment.

3. Review of Theory

The decision rules governing inventory behavior can be thought of as solutions to individual firms' intertemporal profit maximization with constraints. In particular, if we take demand and price to be exogenous, we can express the general problem faced by the firm as the following:

$$Max_Q \; E\left[\sum_{t=0}^{\infty} \beta^t [p_t X_t - f(Q_t) - r_t I_t]\right] \qquad \textbf{(1)}$$

where p_t is the price at time t, X_t is sales at time t, Q_t is quantity produced or purchased, $f(Q_t)$ is the total cost function faced by the firm, I_t is the inventory at time t and r_t is the cost of holding inventory at time t[1] and β is a discount factor.

We can express the total cost function in a general polynomial form as:

$$\begin{aligned} f(Q_t) &= F_0 + \sum_{i=0}^{n} a_i Q_t^{\,i} \quad ; \; if \; Q_t > 0 \\ &= F_0; \quad if \; Q_t = 0 \quad ; \end{aligned} \tag{2}$$

where an n^{th} order polynomial is used to represent the variable cost function. The polynomial can be assumed to be a Taylor series expansion of some nonlinear form around a point.[2] F_0 is the normal fixed costs, and a_0 represents the "quasi-fixed" costs. In the case of production, a_0 would be associated with start up and shutdown of the plant; in the case of purchasing, a_0 is the fixed costs of ordering or delivery. Thus a_0 represents a fixed cost which occurs only when non-zero quantity is produced or ordered in period t.[3] Inventory costs are incurred as long as inventory exists, whether or not production occurs during that period. Inventory at time t is the summation from time 0 to t-1 of the differences between production or purchases and sales and can be written as

[1] There is no distinction made between inventories of finished goods or inventories of materials and supplies. The assumption is that the acquisition of materials and supplies can be factored into costs as a function of quantity produced. For simplicity, however, we can think of inventories as finished goods only, and maintain the homogeneity of product in the problem statement.

[2] Ramey (1989) proposes a polynomial cost function (conditional on quantity and capital) based on input factor prices assuming each category of manufacturing inventory and labor are input factors. There are no such explicit assumptions here.

[3] In some cases this quasi-fixed cost would occur only if production did not occur in period t-1. This problem presents other difficulties and is not considered here.

$$I_t=\sum_{i=0}^{t-1}(Q_i-X_i); \tag{3}$$

Equation 1 can then be written in expanded form as:

$$Max_Q\ E\left[\sum_{t=0}^{\infty}\beta^t[p_tX_t-(F_0+\sum_{i=0}^{n}a_iQ_t^{\,i})-r_t\sum_{j=0}^{t-1}(Q_j-X_j)]\right]; \tag{4}$$
$$where\ \ a_0=0\ \ if\ \ Q_t=0$$

Assuming demand (D_t) is exogenous, we can account for the effect of lost sales by specifying that sales are equal to the lesser of demand and the sum of production and inventory at time t. That is, if demand exceeds the sum of Q_t and I_t, then the difference is lost sales.

Thus

$$X_t= Min(D_t,(Q_t+I_t)) \tag{5}$$

This is the inequality constraint which obtains when orders cannot be back-logged. The penalty for stocking out in this case is the price times the lost sales. More detailed penalty functions try to capture such things as lost customer loyalty by imposing quadratic penalty costs on stockouts. In this case the actual penalty cost is linear, but the expected penalty cost (assuming normally distributed shocks) is convex and non-increasing in inventory level. As inventory levels increase, the probability of stocking out decreases, therefore the expected cost of stocking out falls. At the same time, the probability of inventory exceeding demand increases as inventory increases raising the expected cost of holding inventory. The net result is a "U" shaped inventory cost curve reflecting the summation of a decreasing expected penalty cost from stocking out and an increasing cost of storage as inventory increases. The assumption that demand is a random variable completes the definition of the problem.

Some simplifying assumptions make the problem more tractable. In most cases we assume firms are price takers, which allows us to forego the difficulty of optimizing over both price and quantity. We can also assume that the firm's cost function takes one of two forms. The first is a quadratic (actually convex) cost curve, with a_0 being negligible, which produces production-smoothing motivation in the face of uncertain demand. The second form, which is applicable to the retail and wholesale sectors, is that the fixed cost of ordering, a_0, is significant and the cost function is linear. That is, the marginal cost is constant. This second form with a linear cost function, with particular assumptions regarding the probability distribution of demand, leads to the (S,s) rule. Analogs to this inventory model include the well known Baumol-Tobin money demand model and any model where fixed cost of adjustment and penalty for "stocking out" are traded off against cost of storage.

Proofs of the optimality of (S,s) behavior under various assumptions exist in the literature (see Scarf 1959). In the one period case, recalling that r_t is the cost of storage and p_t is the price of the product, we can determine the penalty cost of stocking out and the cost of having inventory I^*_t at the beginning of period t, assuming orders are filled instantaneously. We assume the firm observes the inventory level at the end of the previous period and then decides the level of inventory to hold for this period. Put another way, the firm makes a decision on what the appropriate level of inventory is for period t, given the level of inventory at the end of the previous period, the expected cost of storage, and the expected cost of stocking out. If demand is a random variable with known probability distribution *g(D)*, and we have an initial inventory level I_{t-1} at the end of period t-1, then we can formulate the expected penalty and holding cost of purchasing sufficient quantity to have an inventory level I^*_t at the beginning of period t as $L(I^*_t)$.

$$L(I^*_t)=p_t\int_{I^*_t}^{\infty}(\xi-I^*_t)g(\xi)d\xi \ + \ r_t\int_{0}^{I^*_t}(I^*_t-\xi)g(\xi)d\xi \tag{6}$$

The first term represents the expected lost revenue from stocking out, which is the expected revenue loss from demand exceeding the inventory level and is a decreasing in function of inventory level. The second term reflects the cost of storage, which is the unit storage cost times the expected excess of inventory over demand and is an increasing function of inventory level. $L(I^*_t)$ is "U" shaped reflecting the sum of the downward sloping and the upward sloping components.

Assuming linear costs of ordering and ignoring fixed costs K_0, from equations 2 and 6, if we order $Q_t = I^*_t - I_{t-1}$ then the total costs (of ordering, stockout and inventory) can be expressed as:

$$f(Q_t) \ + \ L(I^*_t) \ = \begin{cases} a_0+a_1(I^*_t-I_{t-1})+L(I^*_t) & \text{if } I^*_t > I_{t-1} \\ L(I_{t-1}) & \text{if } I^*_t = I_{t-1} \end{cases} \tag{7}$$

Scarf (1959) shows that if we define S as the value of I^*_t that minimizes $a_1I^*_t + L(I^*_t)$ and s as the value of I^*_t which makes

$$a_1s+L(s)=a_0+a_1S+L(S) \tag{8}$$

then it can be shown that the optimal policy is

$$\begin{cases} if \;\; I_{t-1} \geq s, & do \; not \; order \\ if \;\; I_{t-1} < s, & order \; up \; to \; S \end{cases} \tag{9}$$

We can further determine S from the result:

$$G(S) = \frac{p - a_1}{p + r_t} \tag{10}$$

where *G(S)* is the cumulative distribution of the demand.[4] Using the definition in equation 8, we can then obtain the value of s.

The functional determinants of the bandwidth [S,s] are: the distribution of demand, the price of the product, the cost of storage and the "shape" of the cost curve (or more directly, the shape of the marginal cost curve). The interval can be expressed as a function h of these parameters.

$$[S,s] = h(\mu, \sigma, p, r, a_0, a_1, a_2) \tag{11}$$

Note that h(.) is a highly nonlinear and discontinuous function. The important parameters are the relative mark-up between price and marginal cost (which determines the benefit of adjusting) and the probability distribution of demand (which determines the relative cost of storage).

The interest rate elasticity of inventory investment varies depending on the initial conditions. The interest rate affects the bandwidth of desired inventory via the cost of

[4] Note that implicit in this analysis is that there is some level of imperfect competition. If price equals marginal cost then there is no incentive to store. Similarly as storage costs rise, the upper limit S falls.

holding inventory, but the response to a change in interest rate may be unobservable in the contemporaneous change in inventory levels. In other words, a change in [S,s] may or may not result in a contemporaneous change in inventory level depending on the initial location of the firm's inventories within the band. For instance, if S were to rise, but there was sufficient inventory to delay replenishment, no concurrent move would be observed in inventory levels, in fact inventory would fall. If firms needed to replenish at the same time that the desired upper bound increased, inventories would rise above normal.

The upper bound S, increases with:

- an increase in mean demand,
- an increase in the variance of demand,
- a decrease in the cost of storage (r),
- an increase in the mark-up over marginal cost ($p-a_1$).

Similarly, the lower bound s, decreases with:

- an increase in the "quasi-fixed" cost (a_0),
- an increase in the cost of storage (r),
- an increase in the marginal cost,
- a decrease in the price (p),
- a decrease in the mean demand,
- an increase in the variance of demand.

Actual inventory moves in response to demand shocks and initial conditions. In addition, S depends highly on the assumed variance of sales. In the presence of both idiosyncratic and aggregate shocks, firms may be fooled into thinking that the variance of sales increased (i.e.

the aggregate component not recognized as temporary) and adjust the upper limit of their desired inventory level. Thus periods of larger aggregate shocks could result in both synchronization and increased adjustment levels. What is interesting is whether smaller shocks will over time produce full synchronization, absent idiosyncratic redistribution.

4. (S,s) Behaviour as Markov Process

S,s behaviour can be modeled as an n state Markov process where the states are locations within the open interval (s,S] in ℕ. States are represented by discrete (indivisible) levels of inventory. The transition matrix P can be represented by equation 12.

$$P = \begin{array}{cc} & \textit{to state} \\ \begin{array}{c} f \\ r \\ o \\ m \\ \\ s \\ t \\ a \\ t \\ e \end{array} & \begin{bmatrix} 0 & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & 1-\sum_{k=2}^{n-1} p_{1,k} \\ 0 & 0 & p_{1,2} & \cdots & p_{1,n-2} & 1-\sum_{k=2}^{n-2} p_{1,k} \\ .. & .. & .. & \cdots & .. & .. \\ 0 & 0 & 0 & .. & p_{1,2} & 1-p_{1,2} \\ 0 & 0 & 0 & .. & 0 & 1 \\ 1 & 0 & 0 & .. & .. & 0 \end{bmatrix} \end{array} \qquad \textbf{(12)}$$

The elements of P, $\{p_{ij}\}$ are the probabilities of moving from the inventory level in state i to the inventory level in state j after one period. The probability of remaining in a given state after one period is assumed to be zero,[5] and the probability of moving from lower inventory

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[5] Depending on the distribution assumed there can be a non-zero probability of remaining in the same state (i.e. having zero demand), however for this purpose I assume zero. This assumption can be relaxed.

to higher inventory (without going through s_n) is zero while in the interval. State 1 represents the maximum level, S, and state n represents inventory level less than or equal to the minimum, s. Call s_1 the inventory level at S, and s_n inventory level $\leq$ s. Inventory decreases monotonically from s_1 until s_n is reached, at which point it returns to s_1 with certainty (p_{n1}=1.0). The probability of moving from state i to state j ($j \neq n$) depends only on j-i and is independent of the current state, therefore $p_{1j}=p_{k+1,j+k}$ (j,j+k<n). The probability of moving from s_i to s_n in one step is $1-\Sigma p_{ij}$, j<n.

The transition probabilities are a function of the demand distribution which is assumed to be from a discrete distribution. The probability of moving from state i to state j is the probability that the demand will be equal to the difference between the inventory level representing state i and the inventory level at state j. We assume that purchases only occur in moving from state n back to state 1.

$p_{ij} = 0$; $n < j \geq i$,

$p_{n1} = 1.0$; $p_{nj} = 0$; $j > 1$

p_{ij} = Probability that demand equal s_j-s_i, $j \neq n$

p_{in} = Probability that demand is $\geq s_n$-s_i ($i \neq n$)

This generates a transition matrix with zeroes along the diagonal and zeroes everywhere below the diagonal except the first entry of the last row. The elements of the diagonals of submatrices above the main diagonal are equal except for the last column whose elements represent the probability of getting a shock greater than or equal to the amount necessary to lower the inventory to "s".

Now that we know the general look of the transition matrix we can determine the conditions under which the Markov process achieves a "steady state" in the sense that the probability of being in state i after n periods (n very large) is independent of the starting state.[6] The process converges to a steady state for the individual with high probability of being in state one or n and low probability of being in the interim states.

The individual steady state results yield a "U" distribution where the probability of being in the first state or the last state is an order of magnitude higher than any other state. Thus in the long run there is a high probability that an agent will be at the replenishment level or the maximum level than in any other state. The steepness of the "U" will be a function of the width of the (S-s) interval relative to the mean demand. If the interval is close to the mean then the probability of being at S or s in the steady state will be higher compared to being anywhere else. If the demand distribution for each agent is assumed to be independent, then the probability of two agents being in the same state is the inner product of the vector of steady state probabilities of the Markov process. If the probability of being in state 1 or n is high then the inner product will be high. As more agents with independent demand distributions are added, the probability of all of them being in the same state will fall rapidly.

Example:

If we assume a Poisson distribution of demand with mean set to 20 and the S,s interval as (S-s)=60, then the transition matrix will appear as in figure 1, where the y-axis has

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[6]Convergence to a steady state is guaranteed because each row of the transition matrix sums to 1 and one eigenvalue will be 1 and the rest less than one in absolute magnitude.

been truncated at 0.1 for illustration purposes. The probability of going from state i to state i+20 is highest reflecting the distribution. The probability of going from state i to state n

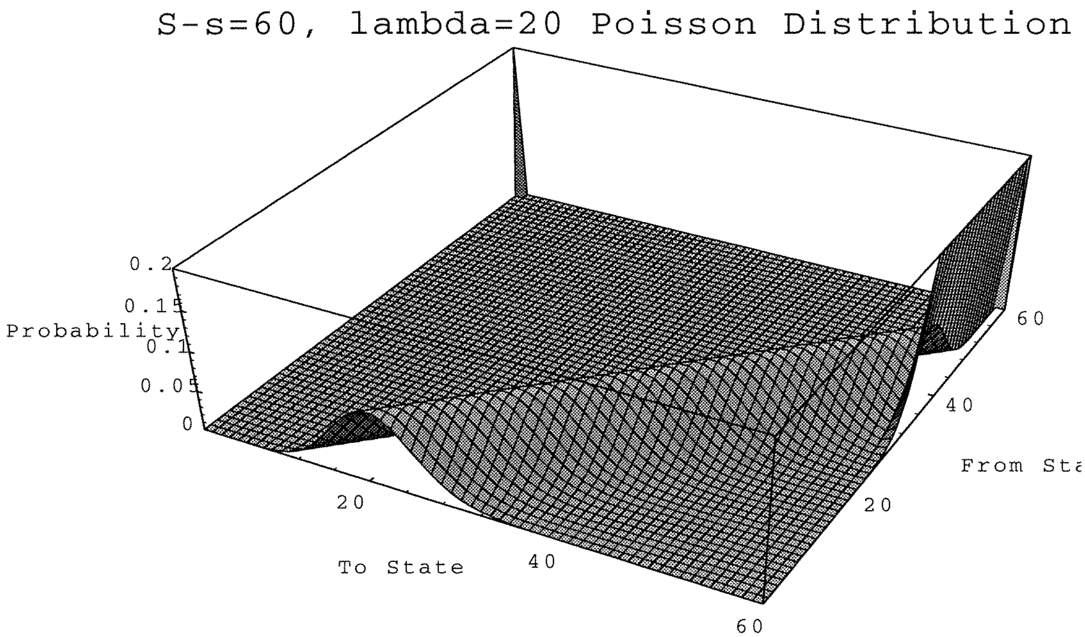

Figure 1 3D representation of transition matrix

increases as i approaches n, becoming 1.0 when i=n-1. The steady state probabilities (which can be obtained by determining the limit of the n period transition matrix as n approaches infinity) yield a 0.225 probability of being in s_1 (S), or s_n (s) versus the next highest probability of being at (S-20) of 0.020, producing the "U" shape discussed above. If we decrease the interval (S-s) to 40 or twice lambda, the steady state probability of being in state 1 or state n is 0.29 and the next highest probability is at state S-20 (0.026). The probability of two agents with the same transition matrix becoming synchronized, assuming independence (or idiosyncratic shocks) is the inner product of the steady state probabilities or 0.109, in the first case and 0.178 in the second. The higher probability of synchronization in the second scenario reflects the higher values of the probabilities of being in states 1 or n. As the number of agents are increased the probability of synchronization will decrease geometrically for independent demand shocks. Aggregate shocks present a more complicated determination of the probability of synchronization because the assumption of independence no longer holds.

Conditions for Synchronization The framework for this analysis is one in which the agent is assumed to replenish to get back exactly to S. Thus whatever the size of the shock which pushes her stock below s, she will purchase enough to return to S. We also temporarily assume agents are identical with identical (S,s) parameters.[7] If an aggregate demand shock greater than or equal to S-s occurs, then all agents will be moved to the state where replenishment is triggered. Thus all agents will synchronize and move together from that point on unless separated by idiosyncratic demand shocks. For all shocks less than S-s, a group of agents within the interval equivalent to the size of the shock above s will move to the replenishment point. For aggregate shocks of constant size, say Δ, agents will become synchronized in (S-s)/Δ groups. For random aggregate shocks, it is sufficient that shocks greater than (S-s)/2 can occur for all agents to become synchronized eventually.

If we start from a large number of agents uniformly distributed in the interval [s,S], and subject them to random aggregate shocks, they will be distributed in groups determined by the first round of shocks summing to more than (S-s). The spacing of these groups will be random, reflecting the differences in the size of the shocks. As long as there are future

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[7] This homogeneity plays a critical part in the continued synchronization under exclusively aggregate shocks. Heterogeneous (S,s) parameters would have the same effect as idiosyncratic shocks in separating agents that have been synchronized by aggregate shocks.

shocks which exceed the spacings between these groups, additional synchronization will occur. In the limit, as long as shocks exceeding (S-s)/2 occur, there is the potential of full synchronization. If the size of the shocks are bounded less than (S-s)/2, then there will be a finite number of discrete groups.

Incorporating the synchronization criteria into the Markov model is not as obvious. The objective is to determine the probability of two (or more) agents being in the same state, given both aggregate and idiosyncratic shocks. Thus the joint probability distribution of both aggregate and idiosyncratic shocks must be used to determine the unconditional probability of both agents moving to the same state.

One option is to determine the probability of the second agent entering that state from all possible states conditional on the first one being there. Thus the probability of an agent moving from state i to state j is the probability that the sum of idiosyncratic and aggregate shocks is equal to the difference between i and j. This is the joint probability distribution of the sum of two random variables.

Another way is to assume a new Markov process consisting of n^2 x n^2 transition matrix reflecting the probability of each agent starting from any given pair of states in the interval and moving to any other given pair of states. The total possible combination of pairs being n^2. The probability of agent 1 being in state i and agent 2 being in state j initially and moving to states k and l respectively can be expressed as

$$P_{kl}^{ij} = p_{ik} \times p_{jl}$$

For coordination purposes we are interested in the case where k=l or when both agents end up in the same state. If shocks are completely aggregate, then this will occur only if i=j or if k=l=n. In other words, since the aggregate shock will be the same to both agents, only occasions where they are both initially at the same position in the inventory interval, or when the shock is large enough to push the agent with the most inventory to the replenishment point, which must also push the agent with less to the replenishment point. If idiosyncratic shocks exist, then agents can end up in the same state, but are just as likely to be separated the next period.

An n^2 x n^2 matrix is computationally cumbersome. It is possible to minimize the computational requirements. For instance, we can take advantage of symmetry as well as the assumption of identical agents. There are n^2 states when we consider both agents simultaneously, but only n of these are of interest, i.e. those where both agents are in the same state. The states can be numbered as 1=(1,1), 2=(1,2), 3=(1,3), ...n=(1,n), n+1=(2,1), n+2=(2,2), n+3=(2,3)....2n=(3,1)......n^2-1=(n,n-1), and n^2=(n,n). The number pair in parentheses represents the location in the interval of each of the two agents. In the case where only aggregate shocks exist, each agent will move by the same amount. Thus non-zero probabilities exist only for cases where movement is from (i,j) to (i+r,j+r), or from (i,j) to (n,n). The case of interest is when agents end up in the same state, starting from different states. This occurs when a shock occurs that is greater than the remaining inventory of the agent with the highest inventory level.

Starting from state 1 the two agents can move to states, n+2 (2,2), 2n+3 (3,3), 3n+4 (4,4), 4n+5 (5,5), (k-1)n+k (k,k) etc. for shocks of 1, 2, 3, 4, and k-1. The probability of moving from state 1 to state n+1 is the probability of demand being 1 unit. Since the shock is aggregate, both agents move by the same amount. This probability is the same as the probability of moving from state 1 to state 2 in the individual. $P_{1,n+2}=p_{1,2}$, and $P_{1,2n+3}=p_{1,3}$... $P_{1,n2}= p_{1,n}$. Starting from state 2 (1,2), agents can move to state n+3 (2,3), state 2n+4 (3,4) .... or n^2 (n,n). The probability of moving from state 2 to state n+3 is the probability of having a demand of 1. $P_{2,n+3}=p_{1,2}$, $P_{2,2n+4}=p_{1,3}$ and so on. This gives a diagonal pattern to the transition matrix with n-1 entries in the first n rows, n-2 in the second n rows and so on.

The symmetry of the matrix is disrupted, however, because unlike the individual transition matrix, there is potential to move "non-monotonically". That is, it is possible to start in a state i and move to a state j|j<i, without first going to state n^2. If the agents start out with in state (k,n) (or (n,k)), then a shock of size ε will move the agents to state (k+ε,1) (or (1,k+ε). For example if the initial state is (10,4) or state n^2 - 6 if n is 10 say, then a shock of 3 will move the agents to state (1,7) or state 7 which will be in the lower triangle. This structure contrasts to the individual case where agents can only move in one direction.

$$P=\begin{bmatrix} 0 & 0 & 0 & \dots & P_{1,n+2} & 0 & 0 & 0 & \dots & P_{1,2n+3} & \dots & 0 & P_{1,n^2} \\ 0 & 0 & 0 & \dots & 0 & P_{2,n+3} & 0 & 0 & \dots & 0 & \dots & 0 & P_{2,n^2} \\ 0 & 0 & 0 & \dots & 0 & 0 & P_{3,n+4} & 0 & \dots & 0 & \dots & 0 & P_{3,n^3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Figure 2 shows the resulting pattern for the transition matix when n=10, i.e. n^2=100, and demand is assumed to be from a Poisson distribution with λ=5. The “from” state is the vertical axis and the "to" state is the horizontal axis. The highest probablitites are centered around states which are separated by shocks of size 5. In other words, from state 1 (where both agents have 10+s units in stock), a shock of size 5 units would move both agents to state 56 where both agents have 5+s units left in stock. Figure 3 is a 3-dimensional representation of the transition matrix.

The steady state probabilities can be computed by checking the convergence of P^k as k gets very large. The computation of the matrix power of 100 x 100 matrix can be tedious. We can approximate by recognizing that when only aggregate shocks obtain, those states where the two agents are at different locations in the interval are transient. These states are nonrecurrent because there is a finite probability that the agents will enter the n^2 state, after which they can never return. The chain can therefore be reduced to just the states where both agents are at the same location (i.e. synchronized). This reduces to an n x n matrix which is more easily maniupulated. In the current example, the 128 step transition using the full matrix, gives a probability of being in state S together as .304, and the probability of being in state s together as .304. The equivalent 128 step transition matrix using the reduced matrix gives the same probabilities.

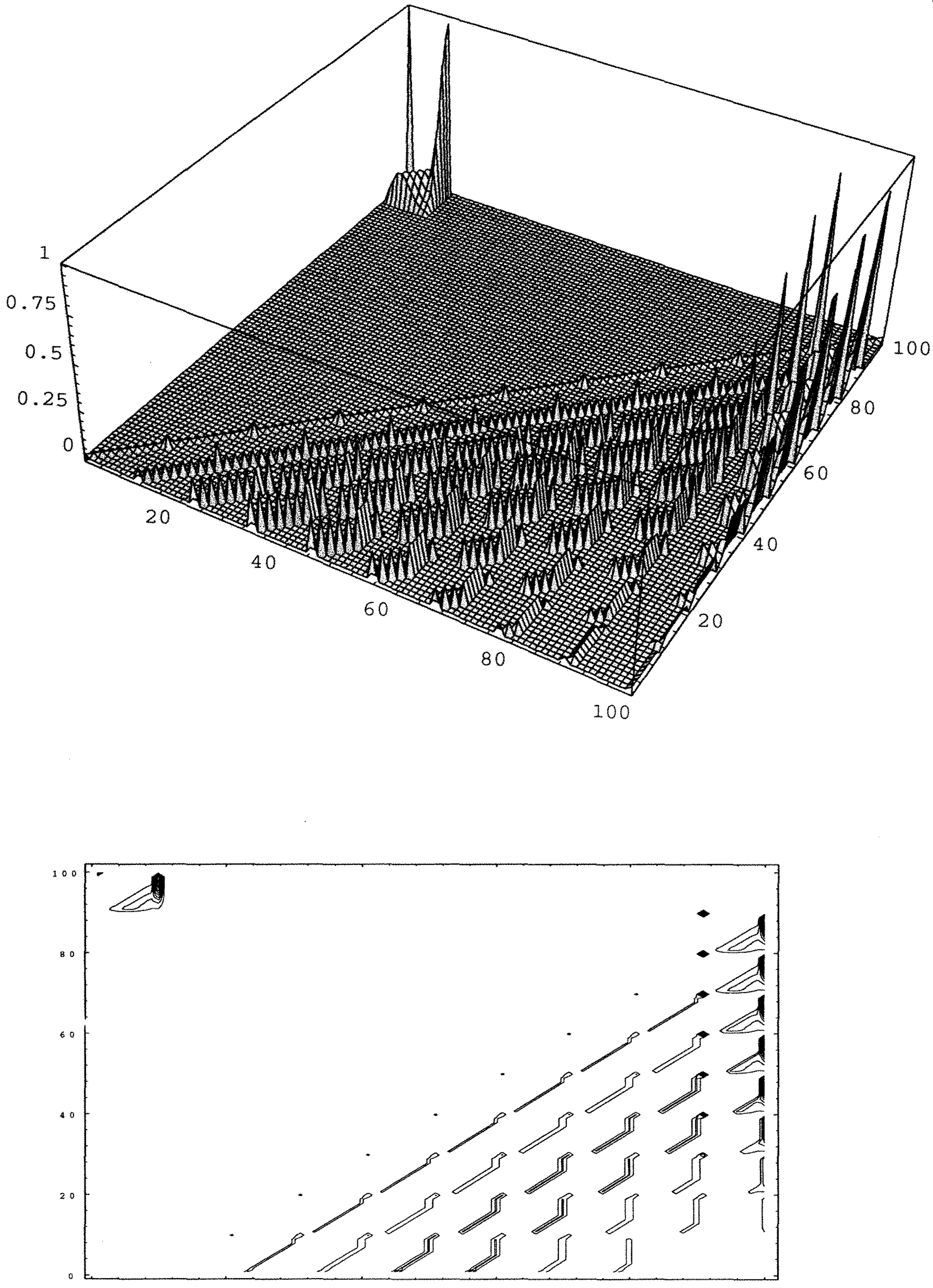

Figure 2 Pattern of transition matrix for two agents

5. Simulations

One hundred identical (S,s) agents subject to aggregate and idiosyncratic demand shocks of different variances over 200 periods were simulated. The resulting aggregate demand, aggregate inventory level and total number of agents adjusting each period was observed to determine the relative time to synchronization. Demand disturbances were assumed normally distributed zero-mean. All agents were initially uniformly distributed in the [S,s] interval. Identical random draws of demand were made for each agent, representing aggregate demand shocks. Idiosyncratic shocks were drawn independently from the same distribution. Inventory was drawn down and replenished subject to the rule that whenever inventory fell below the threshold s, the agent would replenish to the maximum level S. Thus, the only reason for coordination is if agents were pushed below minimum levels simultaneously. The maximum level of inventory, S, was assumed to be 300 units; the minimum level of inventory was assumed to be 100 units. Each firm faces a mean demand of 50 units each period, subject to mean zero aggregate and idiosyncratic shocks. Aggregate shocks of standard deviation 10 and 20 units were assumed and idiosyncratic shocks of zero, 1/4, 1/2, 3/4 and 1 times the standard deviation of the aggregate shocks. The results for each of the ten cases are shown in figures A1-A10.

At a mean demand of 50 units per period and an effective inventory of 200 units, each firm will repelnish on average every 5 periods. As predicted, when faced with exclusively aggregate shocks, all 100 firms eventually synchronize their replenishment schedules. As a result inventory stocks achieve the highest variance, fluctuating from 10,000 units or less, to

30,000 units every 5 periods. As the standard deviation of aggregate shocks increases from 10 to 20, the time to synchronization falls from 100 periods to 40 periods.[8] As idiosyncratic shocks are added, the number of agents replenishing simultaneously falls. When the standard deviation of idiosyncratic shocks are 1/4 of the standard deviation of aggregate shocks, there are occasions when large numbers of agents replenish simultaneously. As the standard deviation of idiosyncratic shocks increases relative to aggregate shocks, the number agents replenishing simultaneously approaches the mean of 20 each period and the aggregate inventory stocks remain close to the mean of 20,000 units. An interesting observation is that the aggregate demand remains relatively unchanged as the idiosyncratic shocks are added. The aggregation of mean-zero shocks over 100 agents effectively cancels out the effect of the idiosyncratic shocks. This effectively makes these idiosyncratic shocks similar to zero-sum idiosyncratic shocks (redistribution among firms) assumed by other authors.

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[8] The random number seed was held constant in these experiments. The actual time to synchronization is of course only a function of the demand draws. The relative time to synchronization reflects the higher probability of a large shock when the variance is higher.

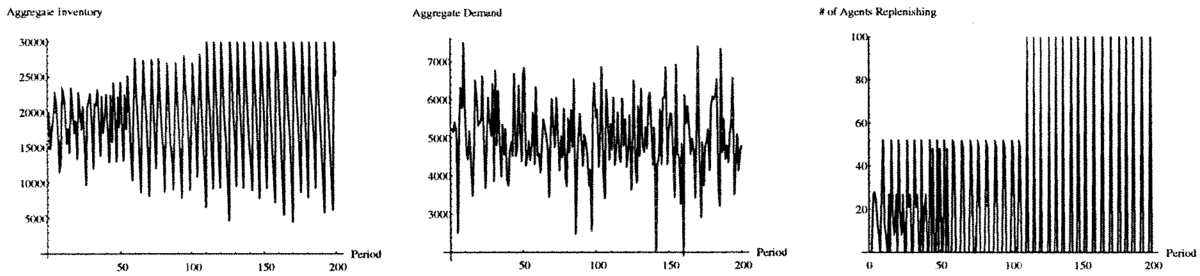
6. Summary and Conclusions

The objective of this paper was to shed some light on the aggregate dynamics of agents who make lumpy decisions. Specifically, when does the lumpy individual behaviour lead to a lumpy aggregate behaviour. The particular agents discussed are agents who follow a (one-sided) (S,s) pattern of adjusting inventory in a discrete-time model. Two approaches to the problem are suggested. First, a Markov model with inventory levels as markov states and transition probabilities based on a Poisson distributed demand was proposed and steady state transition matrices for a single agent and two agents provided some insight. Second, simulations of (S,s) agents facing aggregate and idiosyncratic demand shocks were performed and the aggregate dynamics of inventory stocks were observed.

The use of a discrete Markov model demonstrates the tendency toward synchronization of (S,s) agents in the face of aggregate shocks. The relative variance of the shocks determines the relative time to sychronization of agents. This relative time to synchronization is reflected in the higher steady state probability of being either at the start or end of the inventory interval for higher mean in a Poisson distribution. The simulations using Gaussian sales support the faster time to synchronize at higher variance of (white noise) aggregate shocks. These results confirm the intuition that the relative size of the aggregate shocks will determine the rate of synchronization of lumpy agents.

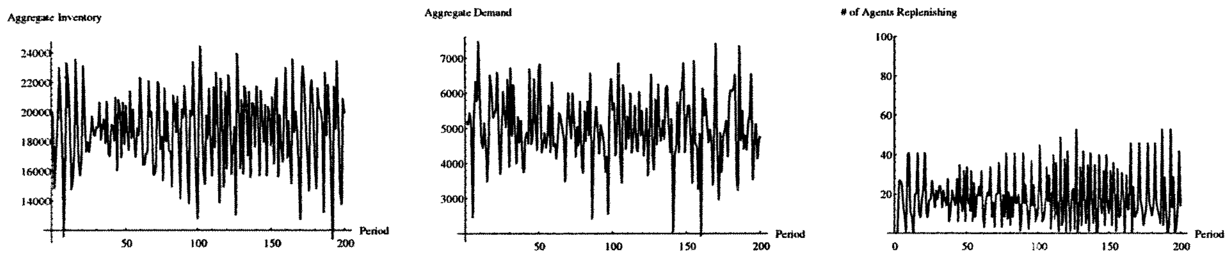
The implications for financial markets are that the distribution of aggregate shocks will determine the rate at which large numbers of lumpy agents synchronize their adjustments. However, any large aggregate shock can result in partial synchronization. This may have occurred in the 1987 stock market adjustment or the 1995 Mexican crisis as an aggregate shock to profit expectations moved many agents to their minimums for adjustment. The synchronization of movement, manifested as herding behaviour resulted in the precipitous decline in asset prices.

The lack of empirical work here is primarily a reflection of the difficulty in extracting relevant parameters on (S,s) behaviour from data at the usual levels of aggregation. Future research into appropriate empirical methods may allow increased capability in predicting turning points in business cycles as well as financial markets.



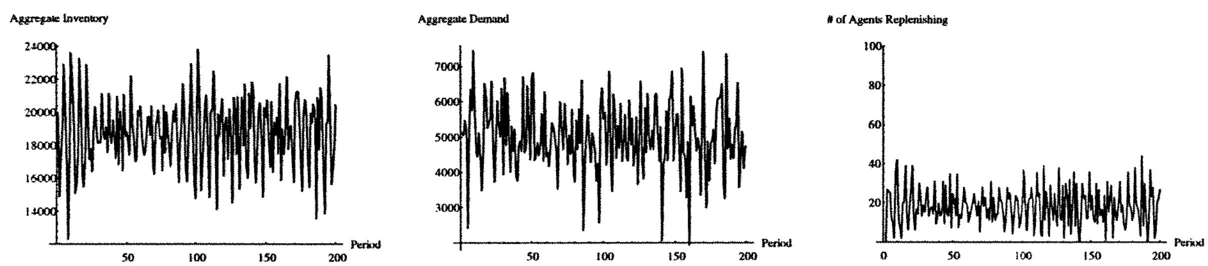
$\sigma_a = 10$
$\sigma_i = 0$

Figure A1 Aggregate shock with σ=10 and no idiosyncratic shocks



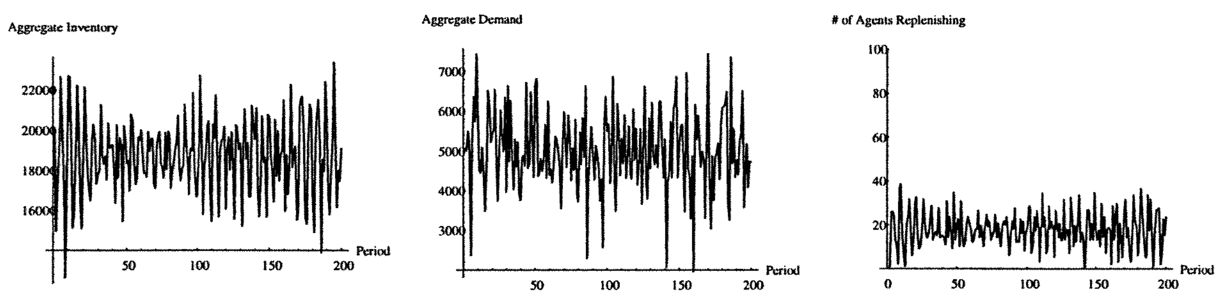
$\sigma_a=10$
$\sigma_i=2.5$

Figure A2 Aggregate shock with σ=10 and idiosyncratic shocks with σ=2.5



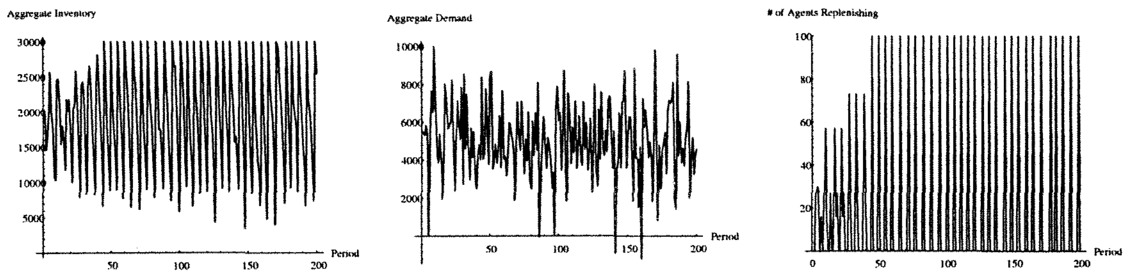
$\sigma_a=10$
$\sigma_i=5$

Figure A3 Aggregate shock with σ=10 and idiosyncratic shocks with σ=5



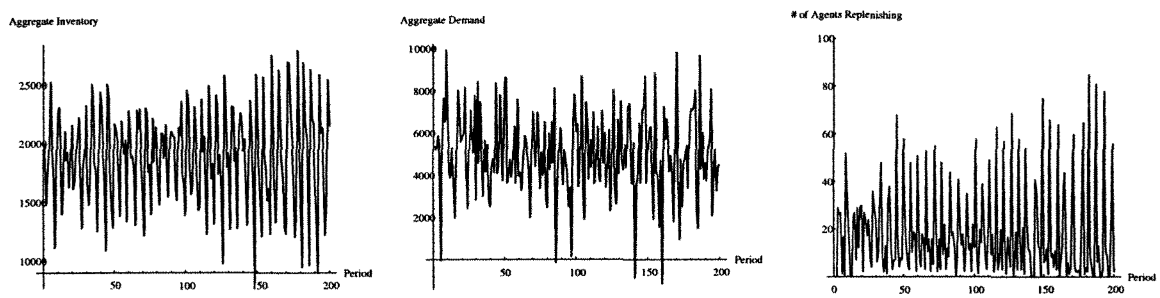
$\sigma_a=10$
$\sigma_i=7.5$

Figure A4 Aggregate shock with σ=10 and idiosyncratic shocks with σ=7.5



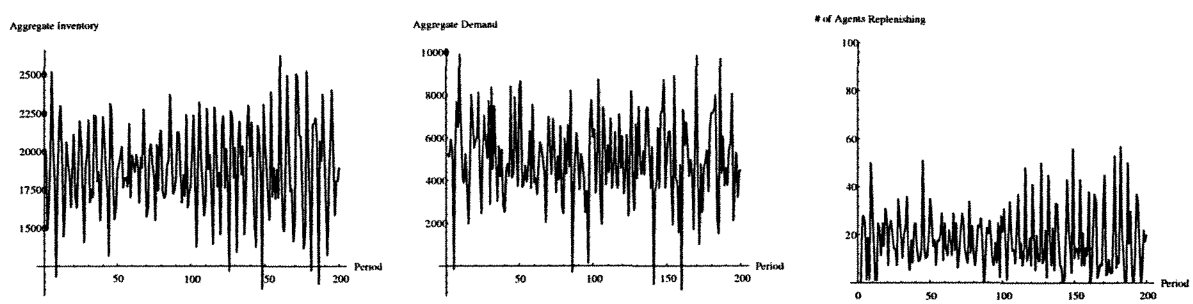
σ_a=20
σ_i=0

Figure A5 Aggregate shocks with σ_a=20 and no idiosyncratic shocks



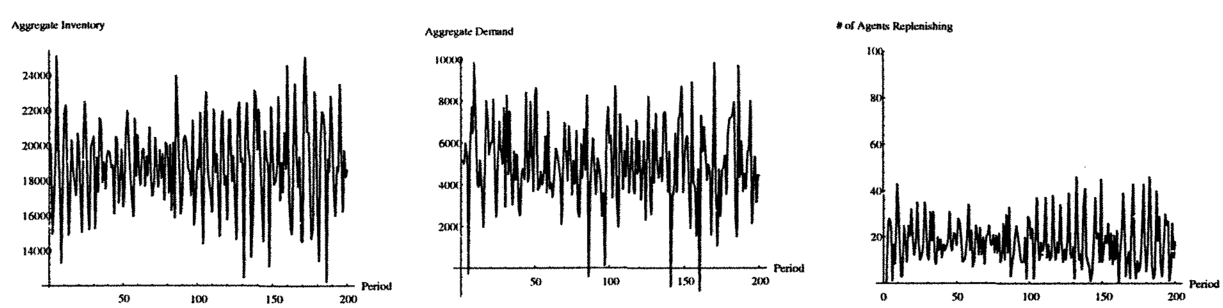
σ_a=20
σ_i=5

Figure A6 Aggregate shocks with σ_a=20 and idiosyncratic shocks with σ_i=5



σ_a=20
σ_i=10

Figure A7 Aggregate shocks with σ_a=20 and idiosyncratic shocks with σ_i=10



σ_a=20
σ_i=15

Figure A8 Aggregate shocks with σ_a=20 and idiosyncratic shocks with σ_i=15

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